Favard length and quantitative rectifiability

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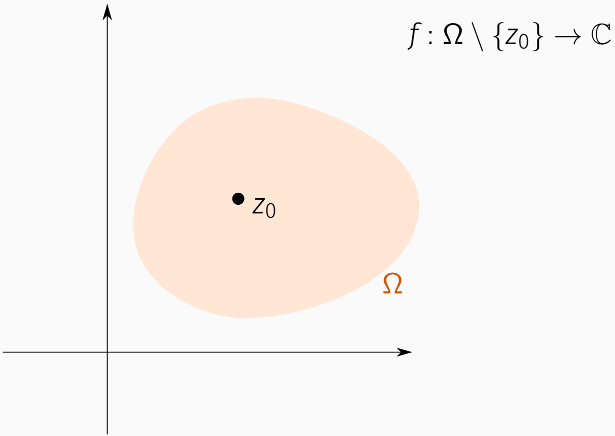
Vitushkin's conjecture

Riemann's theorem on removable singularities

Theorem (Riemann)

If $z_0 \in \Omega \subset \mathbb{C}$ and $f : \Omega \setminus \{z_0\} \to \mathbb{C}$ is analytic and bounded, then f can by extended analytically to all of Ω.



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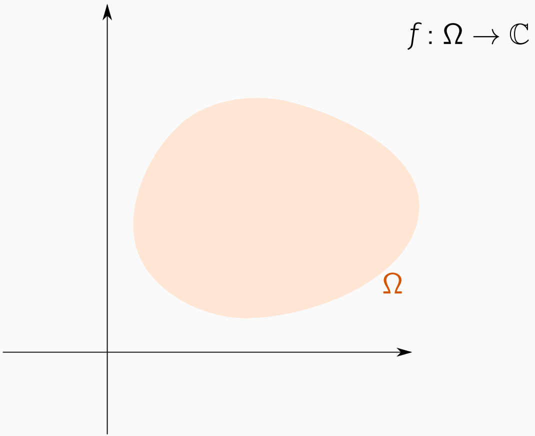
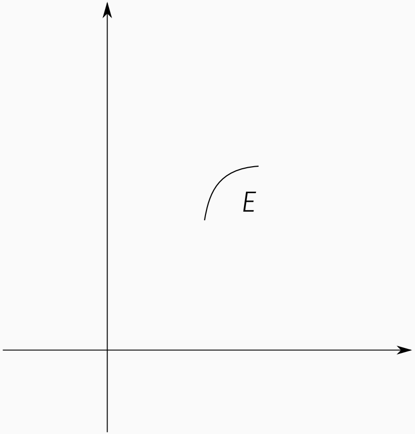


Removable sets

A compact set $E \subset \mathbb{C}$ is **removable for bounded analytic functions** if for any open $\Omega \subset \mathbb{C}$ containing E, each bounded analytic function $f : \Omega \setminus E \to \mathbb{C}$ has an analytic extension to Ω.



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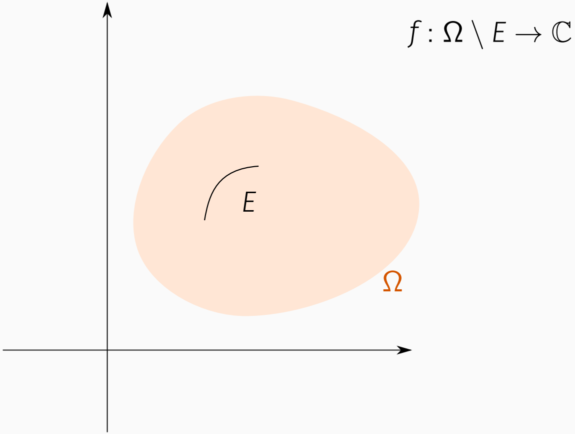

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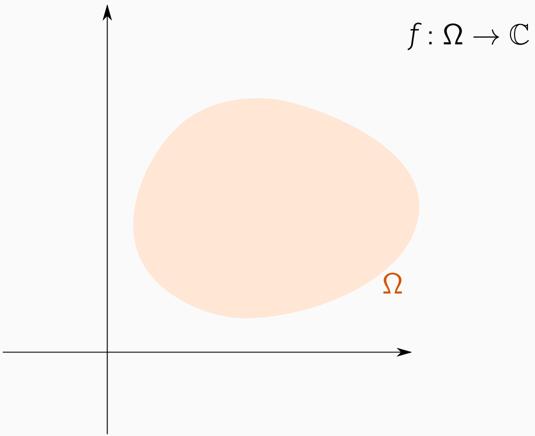

In 1947 Ahlfors characterized removability in terms of **analytic capacity**:

$$E \text{ is removable} \quad \Leftrightarrow \quad \gamma(E) = 0,$$

where

$$\gamma(E) = \sup\{|f'(\infty)| \ : \ f : \mathbb{C} \setminus E \to \mathbb{C} \text{ analytic}, \ \|f\|_\infty \leq 1\},$$
$$f'(\infty) = \lim_{z\to\infty} z(f(z) - f(\infty)).$$

Painlevé problem

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Find a geometric characterization of removable compact sets, i.e. compact sets with $\gamma(E) = 0$.

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Classical:

- If $\mathcal{H}^1(E) = 0$, then $\gamma(E) = 0$.
- If $\dim_H(E) > 1$, then $\gamma(E) > 0$.

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$\gamma(E) = 0 \quad \Leftrightarrow \quad \mathcal{H}^1(E) = 0$?

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Question

$\gamma(E) = 0 \quad \Leftrightarrow \quad \mathcal{H}^1(E) = 0$? No!

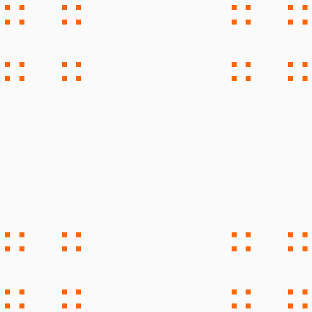
There are sets $E \subset \mathbb{C}$ with $\gamma(E) = 0$ and $0 < \mathcal{H}^1(E) < \infty$.
(Vitushkin 1959, Garnett, Ivanov 1970s)

Vitushkin's conjecture

The sets constructed by Vitushkin, Garnett and Ivanov had very small projections. More precisely, they satisfied

$$\mathcal{H}^1(\pi_\theta(E)) = 0$$

for a.e. direction $\theta \in [0, \pi]$.



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Define **Favard length** of E as

$$\mathrm{Fav}(E) = \int_0^\pi \mathcal{H}^1(\pi_\theta(E))\, d\theta.$$

Vitushkin's conjecture

$$\gamma(E) = 0 \quad \Leftrightarrow \quad \mathrm{Fav}(E) = 0$$

Solution to Vitushkin's conjecture

Vitushkin's conjecture

$$\gamma(E) = 0 \quad \Leftrightarrow \quad \mathrm{Fav}(E) = 0$$

- In the case $\mathcal{H}^1(E) < \infty$ Vitushkin's conjecture is **true**! (Calderón '77, David '98)

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$$\text{Fav}(E) = 0 \quad \not\Rightarrow \quad \gamma(E) = 0.$$

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- What about

$$\mathrm{Fav}(E) = 0 \quad \Leftarrow \quad \gamma(E) = 0?$$

Open problems

Problem 1 (qualitative)

$$\text{Fav}(E) > 0 \quad \Rightarrow \quad \gamma(E) > 0?$$

Open for sets $E \subset \mathbb{C}$ with $\dim_H(E) = 1$ and non-σ-finite $\mathcal{H}^1$-measure.

Open problems

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Problem 2 (quantitative)

$$\gamma(E) \gtrsim \mathrm{Fav}(E)?$$

$$\gamma(E) \gtrsim_{\mathrm{Fav}(E)} 1?$$

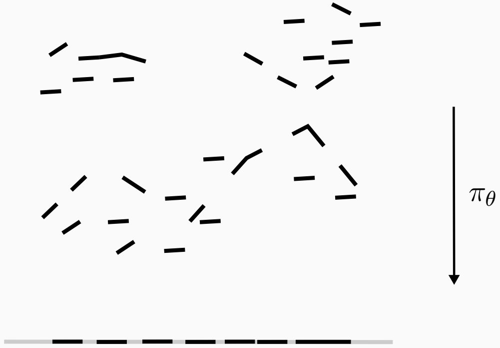
Open even for sets with finite length.

What happens for sets with finite length?

Two ingredients

Theorem (Besicovitch 1939)

Let $E \subset \mathbb{R}^2$ with $0 < \mathcal{H}^1(E) < \infty$. If $\mathrm{Fav}(E) > 0$,



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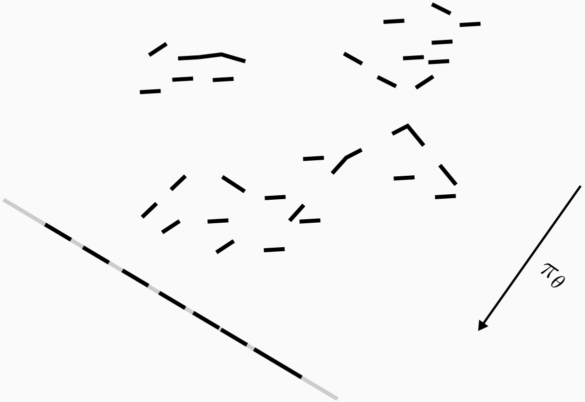
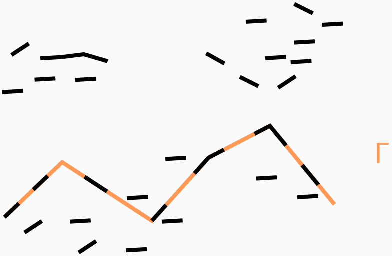


Two ingredients

Theorem (Besicovitch 1939)

Let $E \subset \mathbb{R}^2$ with $0 < \mathcal{H}^1(E) < \infty$. If $\mathrm{Fav}(E) > 0$, then there exists a Lipschitz graph $\Gamma \subset \mathbb{R}^2$ with $\mathcal{H}^1(E \cap \Gamma) > 0$.



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Theorem (Calderón 1977)

If Γ is a rectifiable curve and $F \subset \Gamma$ satisfies $\mathcal{H}^1(F) > 0$, then

$$\gamma(F) > 0.$$

This is a corollary of Calderón's proof of the L^2-boundedness of Cauchy transform on Lipschitz graphs with small constant.

Vitushkin's conjecture when $\mathcal{H}^1(E) < \infty$

Goal

$$\mathrm{Fav}(E) > 0 \quad \Rightarrow \quad \gamma(E) > 0$$

If $0 < \mathcal{H}^1(E) < \infty$ and $\mathrm{Fav}(E) > 0$, then by the Besicovitch projection theorem $\exists\, \Gamma$ with $\mathcal{H}^1(E \cap \Gamma) > 0$

$$\gamma(E) \geq \gamma(E \cap \Gamma)$$

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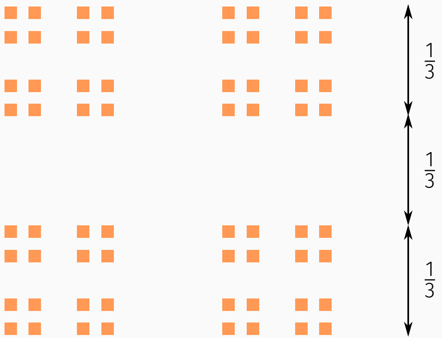
∎

- Why does it only work for sets with finite length?
- Why does it give no quantitative estimates?

First problem

The Besicovitch projection theorem **fails** for sets with infinite length!



$K = C_{1/3} \times C_{1/3}$ satisfies $\mathrm{Fav}(K) \gtrsim 1$ and $\mathcal{H}^1(K \cap \Gamma) = 0$ for every rectifiable curve.

Second problem

Recall: if $0 < \mathcal{H}^1(E) < \infty$ and $\mathsf{Fav}(E) > 0$, then $\exists\, \Gamma$ with $\mathcal{H}^1(E \cap \Gamma) > 0$ and

$$\gamma(E) \geq \gamma(E \cap \Gamma) \overset{\text{(Calderón)}}{>} 0.$$

Second problem

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$$\gamma(E) \geq \gamma(E \cap \Gamma) \overset{\text{(Calderón)}}{>} 0.$$

There are estimates on $\gamma(E \cap \Gamma)$ depending on $\mathcal{H}^1(E \cap \Gamma)$, e.g. if Γ is an L-Lipschitz graph, then

$$\gamma(E \cap \Gamma) \gtrsim_L \mathcal{H}^1(E \cap \Gamma)...$$

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Favard length problem

Can we quantify the dependence of $\mathrm{Lip}(\Gamma)$ and $\mathcal{H}^1(E \cap \Gamma)$ on $\mathrm{Fav}(E)$?

Favard length problem

Naive conjecture...

Theorem (Besicovitch 1939)

Let $E \subset \mathbb{R}^2$ with $0 < \mathcal{H}^1(E) < \infty$. If $\mathsf{Fav}(E) > 0$, then there exists a Lipschitz graph $\Gamma \subset \mathbb{R}^2$ with

$$\mathcal{H}^1(E \cap \Gamma) > 0.$$

Naive conjecture

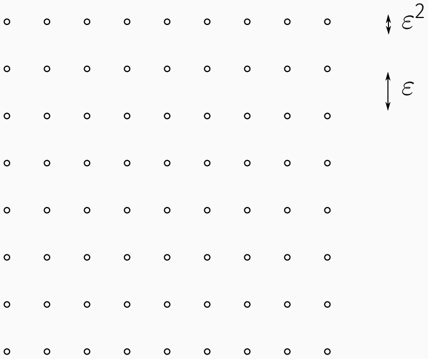
Let $E \subset [0,1]^2$ with $\mathcal{H}^1(E) \sim 1$ and $\mathsf{Fav}(E) \gtrsim 1$. Then, there exists a Lipschitz graph $\Gamma \subset \mathbb{R}^2$ with $\mathsf{Lip}(\Gamma) \lesssim 1$ and

$$\mathcal{H}^1(E \cap \Gamma) \gtrsim 1.$$

… is false

For any $\varepsilon > 0$ there exists a set $E = E_\varepsilon \subset [0,1]^2$ with $\mathcal{H}^1(E) \sim 1$ and $\mathrm{Fav}(E) \gtrsim 1$ such that for all L-Lipschitz graphs Γ

$$\mathcal{H}^1(E \cap \Gamma) \lesssim L\varepsilon.$$

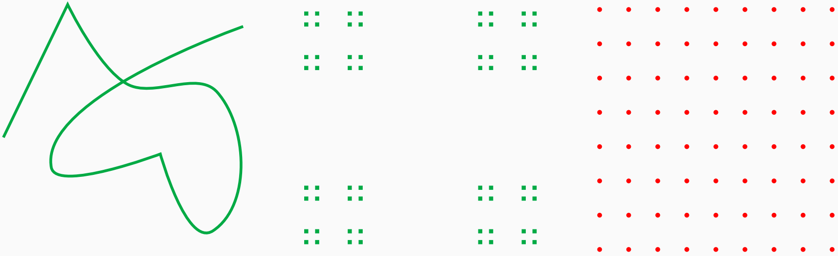


E consists of ε^{-2} uniformly distributed circles of radius ε^2.

Reasonable conjecture

We say that $E \subset \mathbb{R}^2$ is **Ahlfors regular** if for every $x \in E$ and $0 < r < \operatorname{diam}(E)$

$$C^{-1} r \leq \mathcal{H}^1(E \cap B(x,r)) \leq Cr.$$



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Reasonable conjecture

Let $E \subset \mathbb{R}^2$ be an Ahlfors regular set with $\mathrm{Fav}(E) \gtrsim \mathcal{H}^1(E)$.

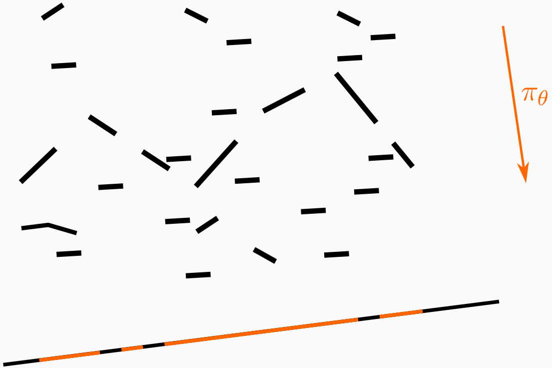
Then, there exists a Lipschitz graph $\Gamma \subset \mathbb{R}^2$ with $\mathrm{Lip}(\Gamma) \lesssim 1$ and

$$\mathcal{H}^1(E \cap \Gamma) \gtrsim \mathcal{H}^1(E).$$

Variations on this conjecture appearing since the 90s in the works of David and Semmes, Mattila, Peres and Solomyak.

What is this really about?

big projections



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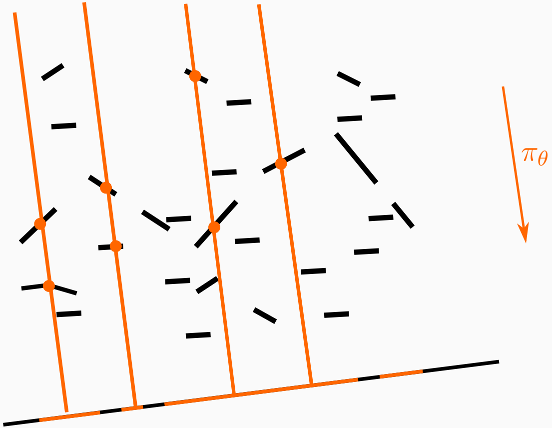
big projections $\Rightarrow$ many lines with few intersections



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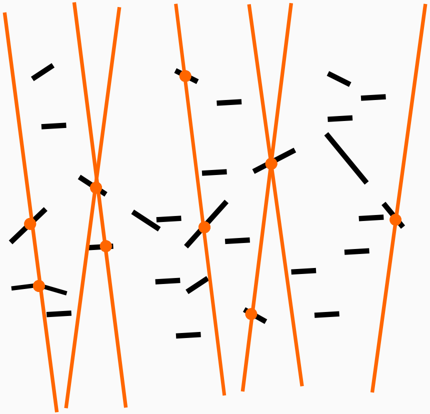

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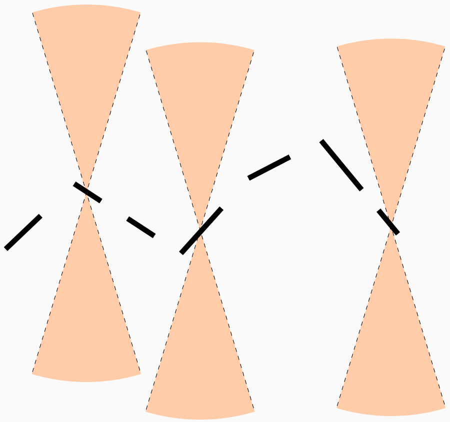

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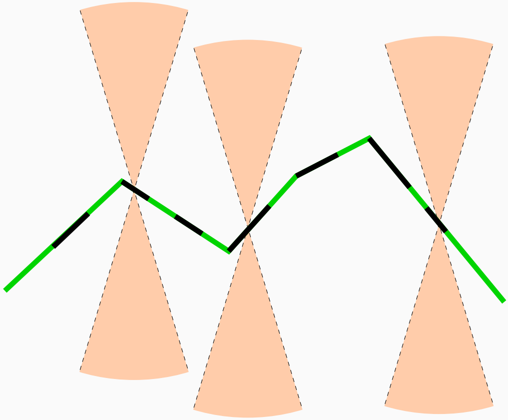
$\Rightarrow$ cones with no intersections $\Rightarrow$ subset of a Lipschitz graph



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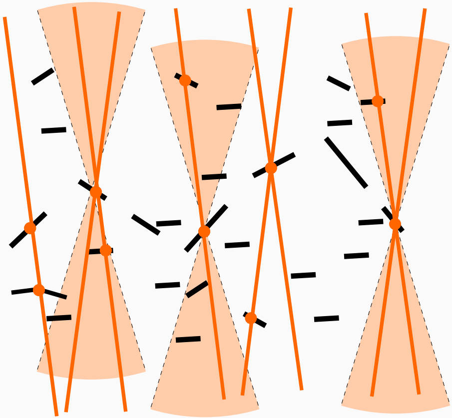

Previous work

Reasonable conjecture

Let $E \subset \mathbb{R}^2$ be an Ahlfors regular set with $\mathrm{Fav}(E) \gtrsim \mathcal{H}^1(E)$.

Then, there exists a Lipschitz graph $\Gamma \subset \mathbb{R}^2$ with $\mathrm{Lip}(\Gamma) \lesssim 1$ and

$$\mathcal{H}^1(E \cap \Gamma) \gtrsim \mathcal{H}^1(E).$$

Progress on the conjecture consisted of replacing "$\mathrm{Fav}(E) \gtrsim \mathcal{H}^1(E)$" by:

- **David-Semmes '93**: big projection + WGL
- **Martikainen-Orponen '18**: projections in L^2
- **Orponen '21**: plenty of big projections
- **D. '22**: projections in L^∞

New result: the conjecture is true!

Theorem (D. '24)

Let $E \subset \mathbb{R}^2$ be an Ahlfors regular set with $\mathsf{Fav}(E) \gtrsim \mathcal{H}^1(E)$.

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Corollaries:

- a positive answer to a 1993 question of David and Semmes,
- a positive answer to a 2002 question of Peres and Solomyak,
- progress on Vitushkin's conjecture.

Back to Vitushkin

Quantitative Vitushkin's conjecture

If $E \subset \mathbb{R}^2$ is compact and $\mathsf{Fav}(E) \geq \kappa\, \mathsf{diam}(E)$, do we have

$$\gamma(E) \gtrsim_\kappa \mathsf{diam}(E)?$$

Partial results in Chang-Tolsa '20 and D.-Villa '22.

Quantitative Vitushkin's conjecture

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Corollary (D. '24)

If $E \subset \mathbb{R}^2$ is Ahlfors regular and $\mathsf{Fav}(E) \geq \kappa\, \mathsf{diam}(E)$, then

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Sets with uniformly large Favard length

We say that a set $E \subset \mathbb{R}^2$ has **uniformly large Favard length** if it is compact and for all $x \in E$ and $0 < r < \operatorname{diam}(E)$

$$\operatorname{Fav}(E \cap B(x,r)) \geq \kappa r.$$

Sets with ULFL

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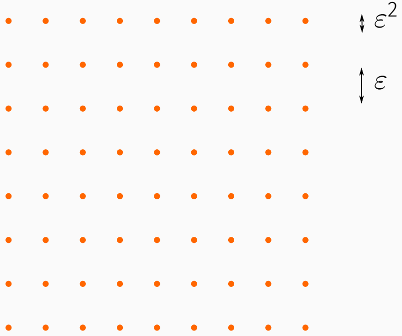

A set violating ULFL

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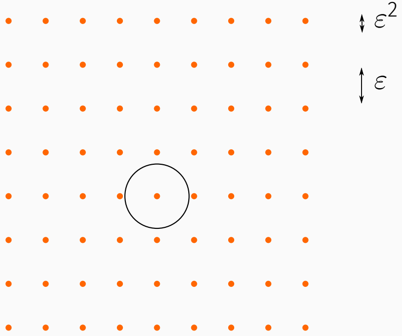

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$$\text{Fav}(E \cap B(x, r)) \geq \kappa r.$$

Corollary (D. '24 + D.-Villa '22)

If $E \subset \mathbb{R}^2$ has ULFL, then

$$\gamma(E) \gtrsim_\kappa \text{diam}(E).$$

Proof of the main result

Goal

Theorem (D. ’24)

Let $E \subset B(0,1)$ be an Ahlfors regular set with $\mathrm{Fav}(E) \gtrsim \mathcal{H}^1(E)$.

Then, there exists a Lipschitz graph $\Gamma \subset \mathbb{R}^2$ with $\mathrm{Lip}(\Gamma) \lesssim 1$ and

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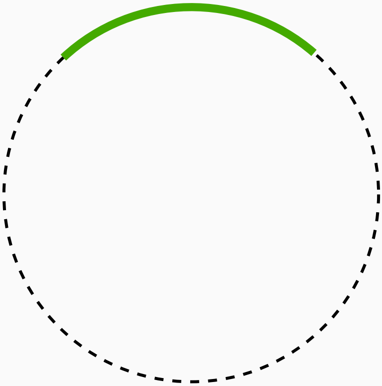
Key tool: conical energies introduced in [Martikainen-Orponen '18] and [Chang-Tolsa '20].

Cones

For any $\theta \in \mathbb{S}^1$ and $x \in \mathbb{R}^2$ set $\ell_{x,\theta} := x + \mathrm{span}(\theta)$.

Given $G \subset \mathbb{S}^1$ and $x \in \mathbb{R}^2$ set

$$X(x, G) := \bigcup_{\theta \in G} \ell_{x,\theta}.$$



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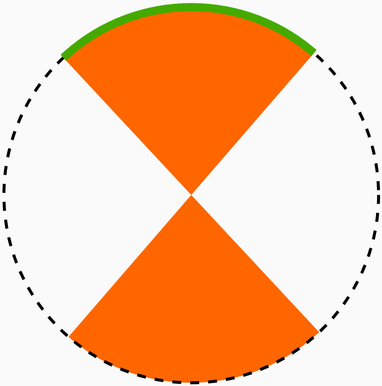
$$X(x, G) := \bigcup_{\theta \in G} \ell_{x,\theta}.$$



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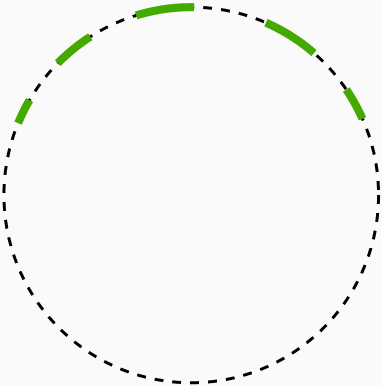

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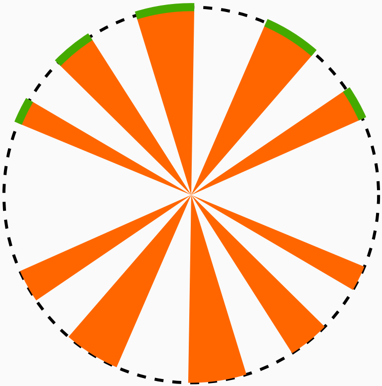

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Given $0 < r < R$ we define the truncated cones

$$X(x, G, r) := X(x, G) \cap B(x, r)$$

and

$$X(x, G, r, R) := X(x, G, R) \setminus B(x, r).$$

Conical energies

Given $x \in \mathbb{R}^2$, $G \subset \mathbb{S}^1$, and a measure μ we define the **conical energy of μ at x** as

$$\mathcal{E}_\mu(x, G) = \int_0^\infty \frac{\mu(X(x, G, r))}{r} \frac{dr}{r}$$

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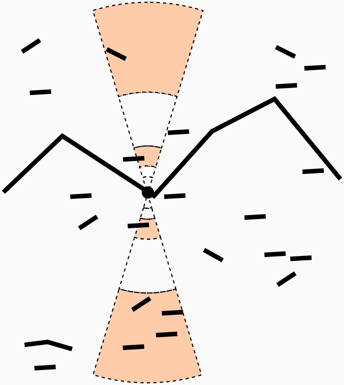
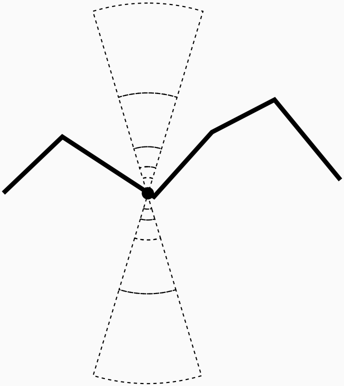


Finding Lipschitz graphs

Note: if $\mathcal{E}_\mu(x, J) = 0$ for μ-a.e. x with a fixed arc $J \subset \mathbb{S}^1$, then

$$\mu(X(x, J)) = 0 \quad \text{for } \mu\text{-a.e. } x,$$

and so μ is concentrated on a Lipschitz graph.



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and so μ is concentrated on a Lipschitz graph.

Theorem (Martikainen-Orponen '18)

Assume that $E \subset B(0, 1)$ is Ahlfors regular, $F \subset E$ with $\mathcal{H}^1(F) \sim \mathcal{H}^1(E)$, and there exists an arc $J \subset \mathbb{S}^1$ with $\mathcal{H}^1(J) \gtrsim 1$ such that for $\mu = \mathcal{H}^1|_F$

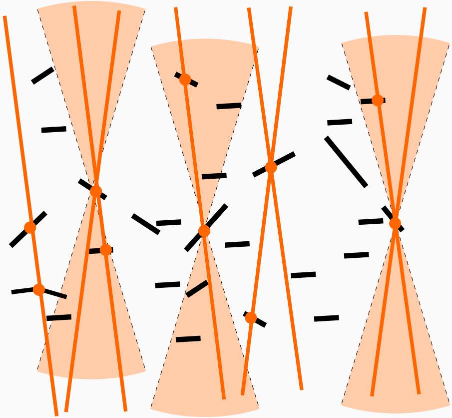
$$\mathcal{E}_\mu(x, J) \lesssim 1.$$

Then, there exists a Lipschitz graph $\Gamma \subset \mathbb{R}^2$ with $\mathrm{Lip}(\Gamma) \lesssim 1$ and $\mathcal{H}^1(F \cap \Gamma) \gtrsim 1$.

From big projections to conical energies

big projections $\Rightarrow$ many lines with few intersections

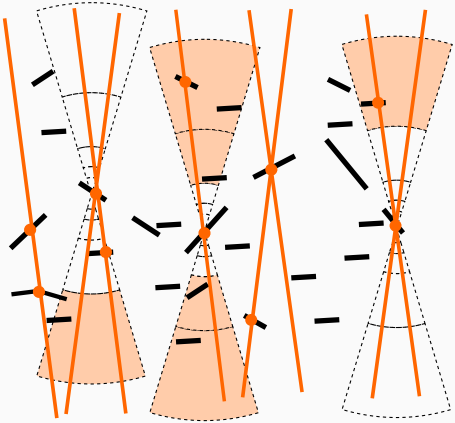
$\Rightarrow$ cones with no intersections $\Rightarrow$ subset of a Lipschitz graph



From big projections to conical energies

big projections $\Rightarrow$ many lines with few intersections

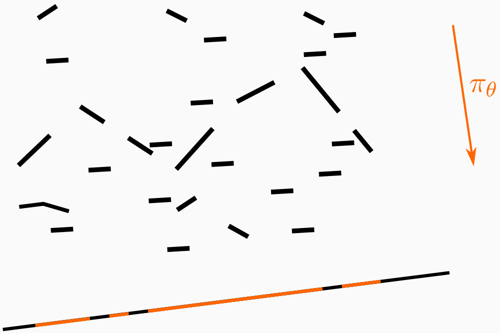
$\Rightarrow$ bounded conical energies $\overset{\text{[MO18]}}{\Rightarrow}$ subset of a Lipschitz graph



"Many lines with few intersections"

Lemma

Let $E \subset B(0,1)$ be an Ahlfors regular set with $\mathrm{Fav}(E) \gtrsim \mathcal{H}^1(E)$.



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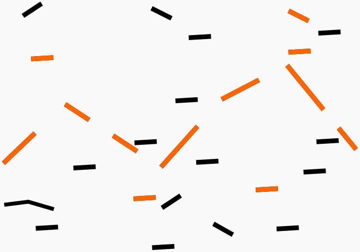

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Lemma

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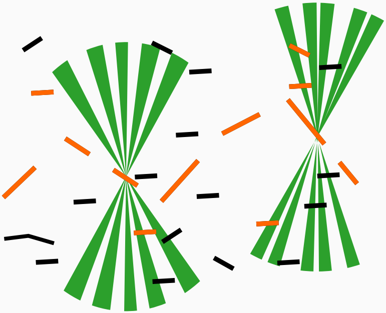

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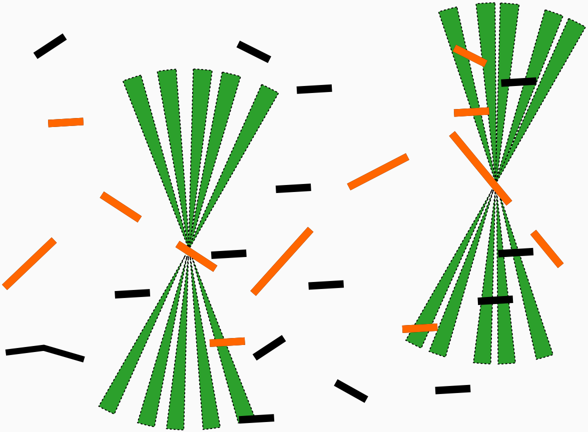
This is close to [MO18], but there are two problems:

- $G(x) \subset \mathbb{S}^1$ might not be an arc,
- $G(x)$ depends on the point x.

[MO18] requires that $G(x) = J$ for some fixed arc $J \subset \mathbb{S}^1$.

Good directions propagate

$$\int_F \mathcal{E}_\mu(x, G(x))\, d\mu(x) \lesssim \mu(F).$$



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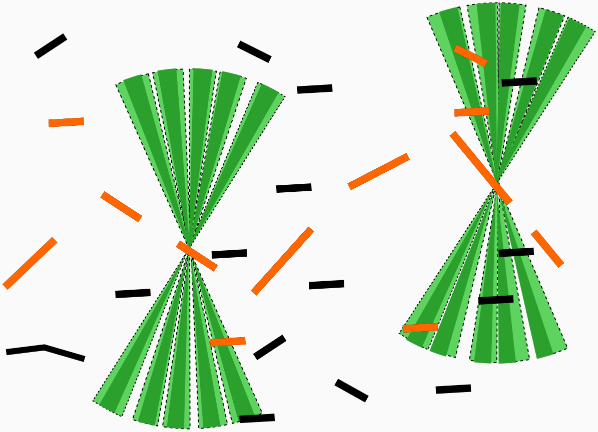
$$\int_F \mathcal{E}_\mu(x, G_*(x))\, d\mu(x) \lesssim \int_F \mathcal{E}_\mu(x, G(x))\, d\mu(x) \lesssim \mu(F).$$



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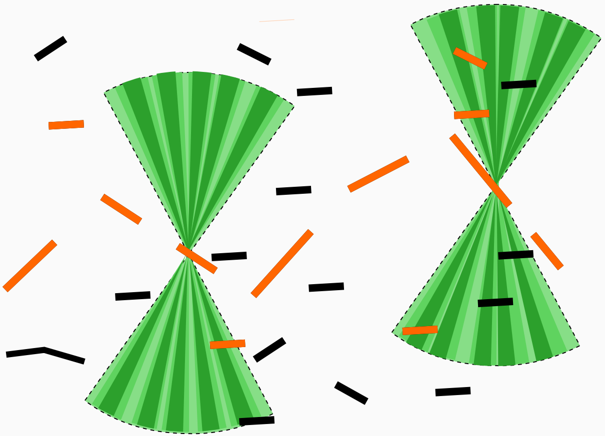

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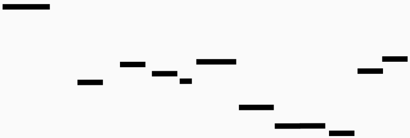
Proof of the main result:

Lemma + Propagation + [MO18] = big piece of a Lipschitz graph.

Toy version of the propagation result

Proposition

Let $E \subset B(0,1)$ be an Ahlfors regular set consisting of parallel segments.



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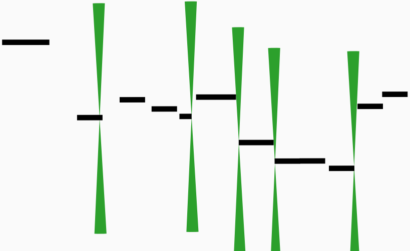

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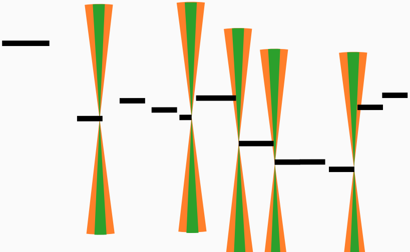

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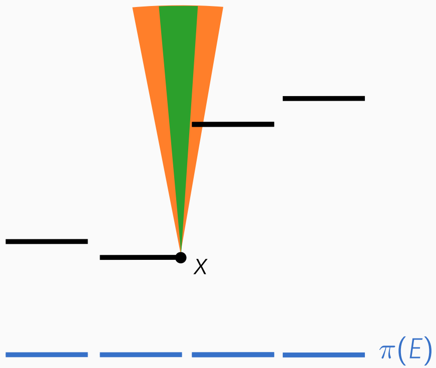
$$\mathcal{E}_\mu(x, 3J) = \mathcal{E}_\mu(x, 3J \setminus J) \sim \sum_{k \in \mathbb{Z}} \frac{\mu(X(x, 3J \setminus J, 2^{-k}, 2^{-k+1}))}{2^{-k}}$$

$$= \sum_{k \in \mathrm{Bad}(x)} \frac{\mu(X(x, 3J \setminus J, 2^{-k}, 2^{-k+1}))}{2^{-k}} \sim \mathcal{H}^1(J) \cdot \#\mathrm{Bad}(x).$$

Key geometric lemma

If $k \in \mathrm{Bad}(x)$, then there exists a "gap" I in $\pi(E)$ such that

$$\mathcal{H}^1(I) \sim \mathcal{H}^1(J) \cdot 2^{-k} \quad \text{and} \quad \pi(x) \in 5I.$$



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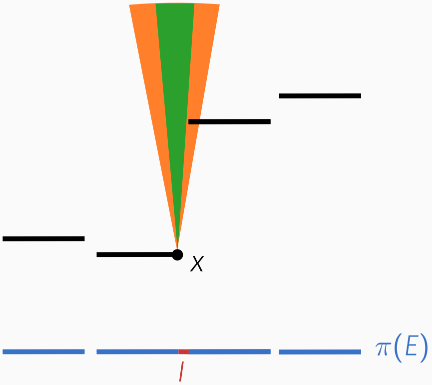

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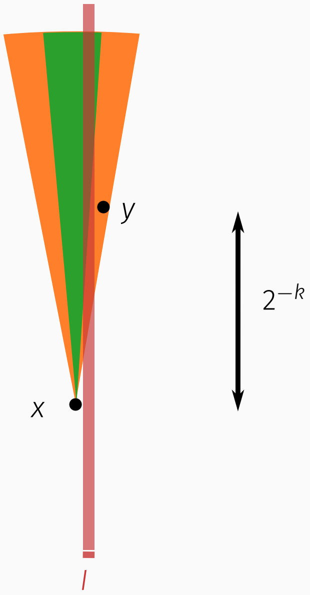
&\sim \mathcal{H}^1(J) \sum_{k \geq 0} \sum_{\substack{I \in \mathrm{Gap}, \\ \mathcal{H}^1(I) \sim \mathcal{H}^1(J) 2^{-k}}} \mathcal{H}^1(I) \lesssim \mathcal{H}^1(J)\, \mathrm{diam}(E).
\end{aligned}$$

“Proof” of the key geometric lemma



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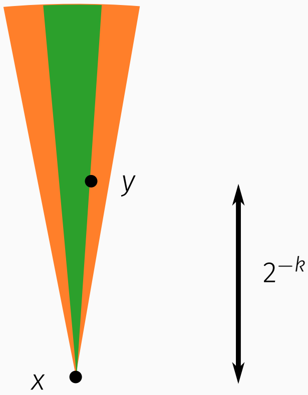

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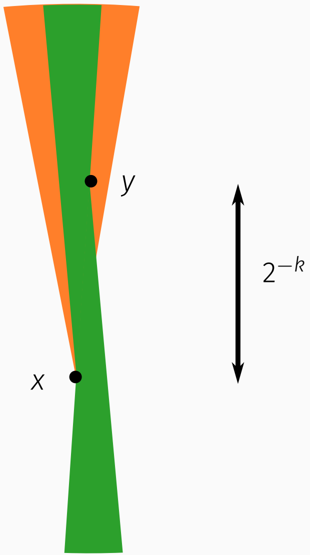

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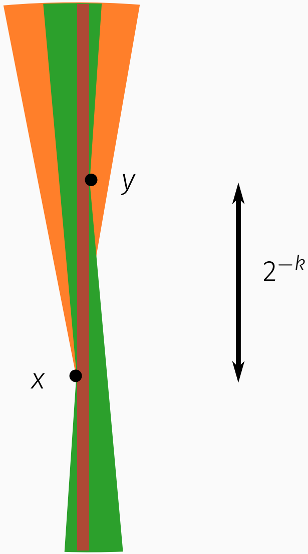

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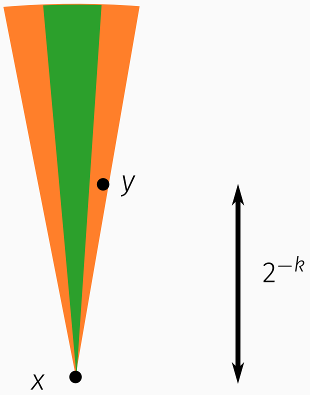

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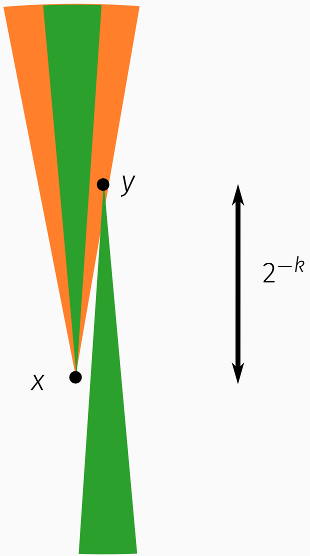

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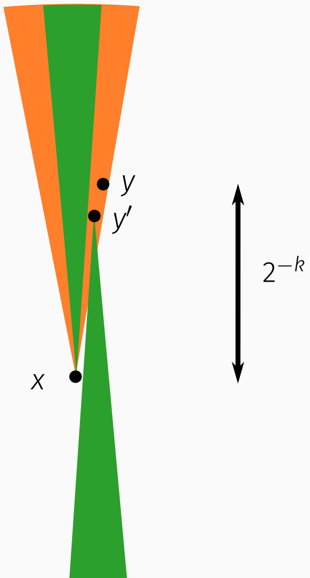

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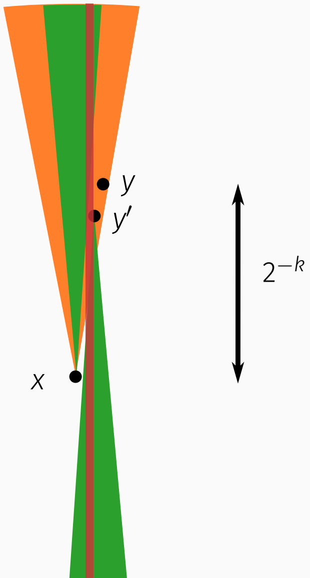

A question

Qualitative ULFL

Suppose that E is compact, and for every $x \in E$ we have

$$\liminf_{r \to 0} \frac{\mathrm{Fav}(E \cap B(x,r))}{r} > 0.$$

Does this imply $\gamma(E) > 0$?

Qualitative ULFL

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Thank you!