# Favard length problem for Ahlfors regular sets

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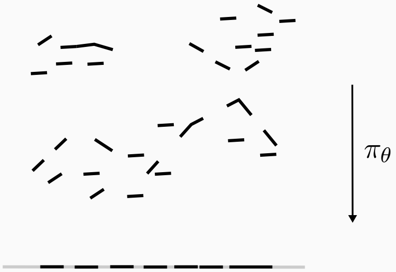
# Besicovitch projection theorem

Favard length of $E \subset \mathbb{R}^2$ is

$$\mathrm{Fav}(E) = \int_0^\pi \mathcal{H}^1(\pi_\theta(E))\, d\theta.$$

**Theorem (Besicovitch 1939)**

Let $E \subset \mathbb{R}^2$ with $0 < \mathcal{H}^1(E) < \infty$. If $\mathrm{Fav}(E) > 0$,



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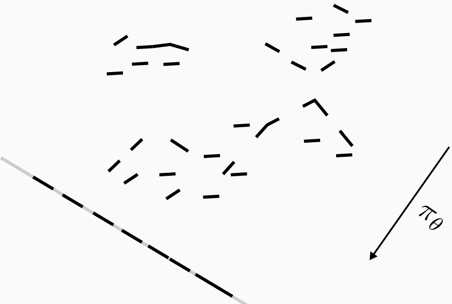

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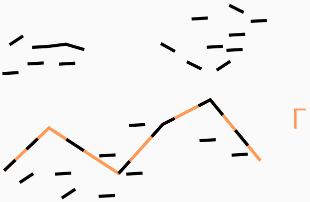
Let $E \subset \mathbb{R}^2$ with $0 < \mathcal{H}^1(E) < \infty$. If $\mathsf{Fav}(E) > 0$, then there exists a Lipschitz graph $\Gamma \subset \mathbb{R}^2$ with $\mathcal{H}^1(E \cap \Gamma) > 0$.



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**Favard length problem**

Can we quantify the dependence of $\mathrm{Lip}(\Gamma)$ and $\mathcal{H}^1(E \cap \Gamma)$ on $\mathrm{Fav}(E)$?

# Naive conjecture...

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**Naive conjecture**

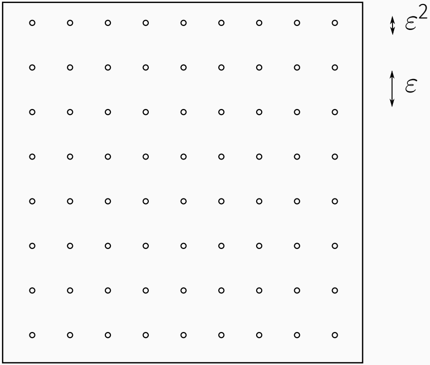
Let $E \subset [0,1]^2$ with $\mathcal{H}^1(E) \sim 1$ and $\mathrm{Fav}(E) \gtrsim 1$. Then, there exists a Lipschitz graph $\Gamma \subset \mathbb{R}^2$ with $\mathrm{Lip}(\Gamma) \lesssim 1$ and

$$\mathcal{H}^1(E \cap \Gamma) \gtrsim 1.$$

# … is false

For any $\varepsilon > 0$ there exists a set $E = E_\varepsilon \subset [0,1]^2$ with $\mathcal{H}^1(E) \sim 1$ and $\mathrm{Fav}(E) \gtrsim 1$ such that for all $L$-Lipschitz graphs $\Gamma$

$$\mathcal{H}^1(E \cap \Gamma) \lesssim L\varepsilon.$$




$E$ consists of $\varepsilon^{-2}$ uniformly distributed circles of radius $\varepsilon^2$.

# Reasonable conjecture

We say that $E \subset \mathbb{R}^2$ is **Ahlfors regular** if for every $x \in E$ and $0 < r < \text{diam}(E)$

$$C^{-1} r \leq \mathcal{H}^1(E \cap B(x, r)) \leq Cr.$$

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**Reasonable conjecture**

Let $E \subset \mathbb{R}^2$ be an Ahlfors regular set with $\mathrm{Fav}(E) \gtrsim \mathcal{H}^1(E)$.

Then, there exists a Lipschitz graph $\Gamma \subset \mathbb{R}^2$ with $\mathrm{Lip}(\Gamma) \lesssim 1$ and

$$\mathcal{H}^1(E \cap \Gamma) \gtrsim \mathcal{H}^1(E).$$

# Previous work

**Reasonable conjecture**

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Progress on the conjecture consisted of replacing "$\mathrm{Fav}(E) \gtrsim \mathcal{H}^1(E)$" by:

- **David-Semmes '93**: big projection + WGL
- **Martikainen-Orponen '18**: projections in $L^2$
- **Orponen '21**: plenty of big projections
- **D. '22**: projections in $L^\infty$

# New result: the conjecture is true!

**Theorem (D. '24)**

Let $E \subset \mathbb{R}^2$ be an Ahlfors regular set with $\mathrm{Fav}(E) \geq \kappa \mathcal{H}^1(E)$.

Then, there exists a Lipschitz graph $\Gamma \subset \mathbb{R}^2$ with $\mathrm{Lip}(\Gamma) \lesssim_\kappa 1$ and

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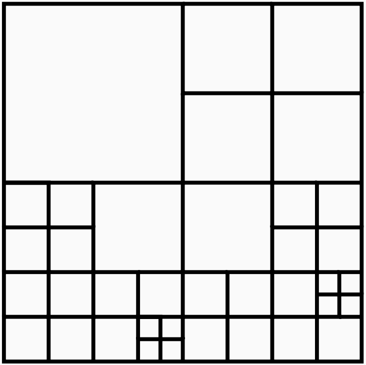
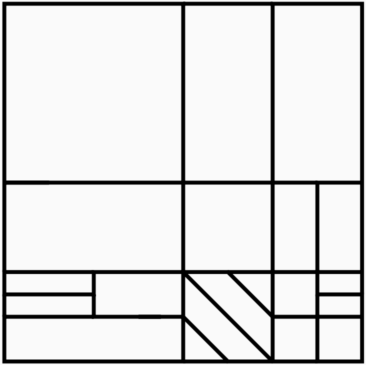
Remarks:

- explicit dependence on $\kappa$
- the proof likely works in higher dimensions

# About the proof

- main tool: conical energies of Chang-Tolsa; continuation of D. ’22
- key novelty: multiscale decomposition involving scales, locations, and directions:



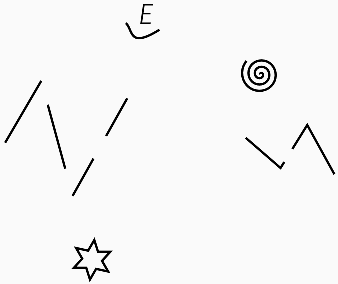


# Corollary: David-Semmes question

# Big pieces of Lipschitz graphs

An Ahlfors regular set $E$ contains **big pieces of Lipschitz graphs** if there exist $C, L > 0$ such that for every $x \in E$ and every $0 < r < \mathrm{diam}(E)$ there exists an $L$-Lipschitz graph $\Gamma = \Gamma_{x,r}$ with

$$\mathcal{H}^1(E \cap \Gamma \cap B(x, r)) \geq Cr.$$



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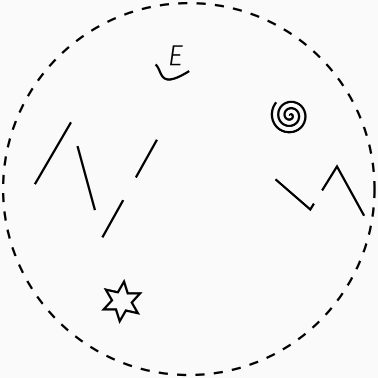

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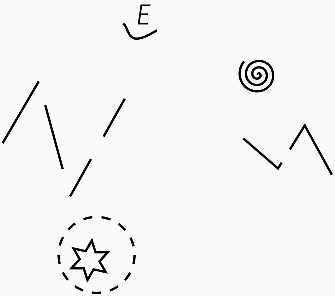

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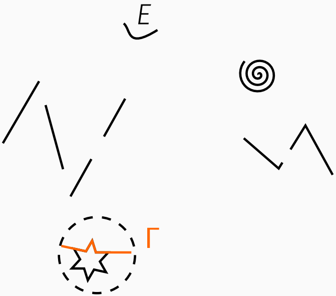

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**Question (David-Semmes '93)**

Let $E \subset \mathbb{R}^2$ be an Ahlfors regular set such that for every $x \in E$ and $0 < r < \operatorname{diam}(E)$ we have $\operatorname{Fav}(E \cap B(x,r)) \gtrsim r$.

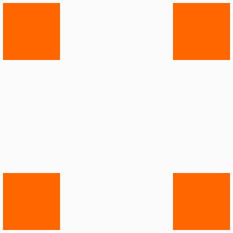
Does $E$ contain big pieces of Lipschitz graphs?

**Corollary (D. '24)**

Yes it does! Thus, ULFL $\Leftrightarrow$ BPLG.

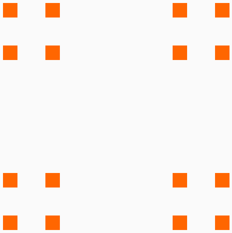
# Corollary: Peres-Solomyak question

# Peres-Solomyak question



$K_1$

# Peres-Solomyak question



$K_2$

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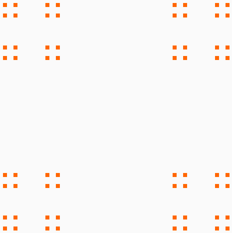

$K_3$

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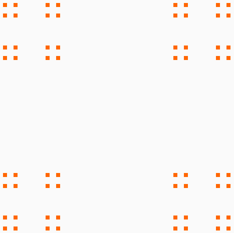

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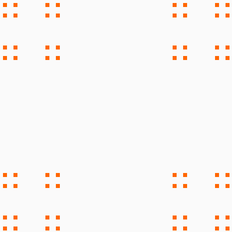

$$K = \bigcap_n K_n$$

**Question (Peres-Solomyak '02)**

What is the rate of decay of

$$\mathrm{Fav}(K_n) \xrightarrow{n\to\infty} 0?$$

What about more general purely unrectifiable sets?

# Quantifying pure unrectifiability

- Recall: $E \subset \mathbb{R}^2$ is **purely unrectifiable** if for every rectifiable curve $\Gamma$ we have $\mathcal{H}^1(E \cap \Gamma) = 0$.

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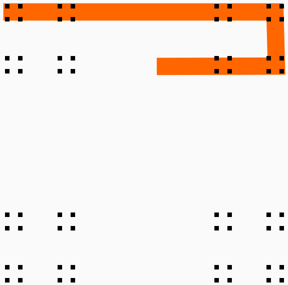
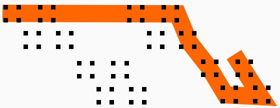
- Consider
$$\ell(E,\delta) = \sup_{\Gamma} \mathcal{H}^1_\infty(E \cap \Gamma(\delta))$$
with supremum taken over curves $\Gamma$ with $\mathcal{H}^1(\Gamma) = \mathrm{diam}(E)$.





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with supremum taken over curves $\Gamma$ with $\mathcal{H}^1(\Gamma) = \operatorname{diam}(E)$.
- For purely unrectifiable sets with $0 < \mathcal{H}^1(E) < \infty$ we have
$$\operatorname{Fav}(E(\delta)) \xrightarrow{\delta \to 0} 0 \quad \text{and} \quad \ell(E, \delta) \xrightarrow{\delta \to 0} 0.$$

**Question (Peres-Solomyak '02)**

Can one estimate $\operatorname{Fav}(E(\delta))$ in terms of $\ell(E, \delta)$?

# Previous work

If $E$ is self-similar or random, there are plenty of estimates for $\mathrm{Fav}(E(\delta))$:

Peres-Solomyak '02, Tao '09, Łaba-Zhai '10, Bateman-Volberg '10, Nazarov-Peres-Volberg '11, Bond-Łaba-Volberg '14, Bond-Łaba-Zahl '14, Wilson '17, Bongers '19, Cladek-Davey-Taylor '20, Bongers-Taylor '21, Łaba-Marshall '22, Davey-Taylor '22, Vardakis-Volberg '24...

In general, no estimate for $\mathrm{Fav}(E(\delta))$ in terms of $\ell(E, \delta)$.

# New estimate

**Corollary (D. '24)**

If $E \subset \mathbb{R}^2$ is Ahlfors regular, then

$$\mathrm{Fav}(E(\delta)) \leq \frac{C_\epsilon}{(\log\log\log(\ell(E,\delta)^{-1}))^{1/3-\epsilon}}.$$

- For the 4-corners Cantor set:

$$\mathrm{Fav}(K_n) \leq \frac{C_\epsilon}{(\log\log\log n)^{1/3-\epsilon}}.$$

  State of the art is [Nazarov-Peres-Volberg '11]:

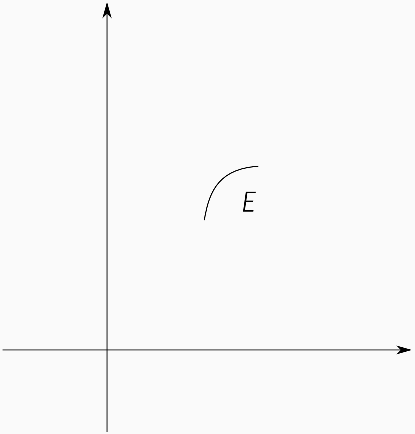
$$\mathrm{Fav}(K_n) \leq \frac{C}{n^c}.$$

- No self-similarity needed!

# Corollary: Vitushkin's conjecture

# Removable sets

A compact set $E \subset \mathbb{C}$ is **removable for bounded analytic functions** if for any open $\Omega \subset \mathbb{C}$ containing $E$, each bounded analytic function $f : \Omega \setminus E \to \mathbb{C}$ has an analytic extension to $\Omega$.



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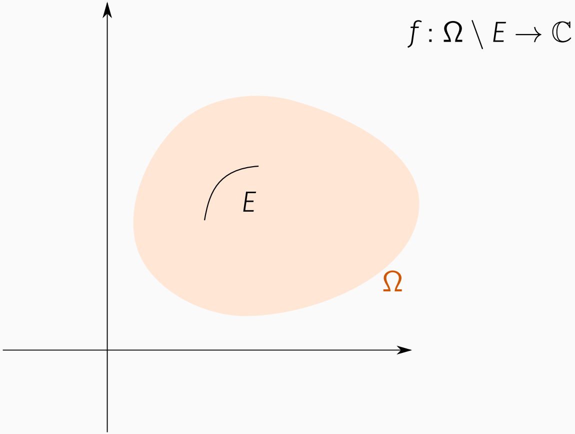

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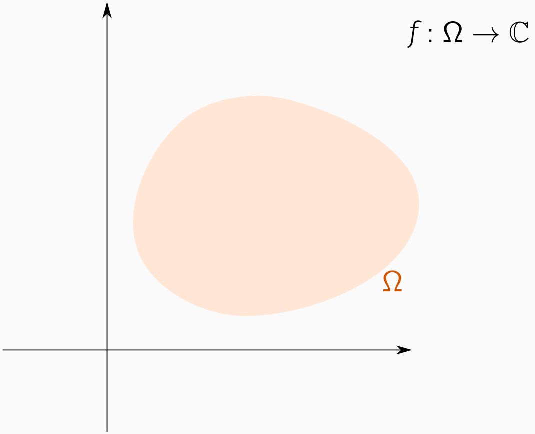

# Analytic capacity

In 1947 Ahlfors characterized removability in terms of analytic capacity:

$$E \text{ is removable} \quad \Leftrightarrow \quad \gamma(E) = 0,$$

where

$$\gamma(E) = \sup\{|f'(\infty)| \;:\; f : \mathbb{C} \setminus E \to \mathbb{C} \text{ analytic}, \; \|f\|_\infty \leq 1\},$$
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- If $\dim_H(E) < 1$ or $\dim_H(E) > 1$ Vitushkin's conjecture is **true**! (easy)

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- What about

$$\mathrm{Fav}(E) = 0 \quad \Leftarrow \quad \gamma(E) = 0?$$

# Quantitative Vitushkin's conjecture

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If $E \subset \mathbb{R}^2$ is compact and $\mathsf{Fav}(E) \geq \kappa\, \mathsf{diam}(E)$, do we have

$$\gamma(E) \gtrsim_\kappa \mathsf{diam}(E)?$$

Partial results in Chang-Tolsa '20 and D.-Villa '22.

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## Corollary (D. + D.-Villa '22)

If $E \subset \mathbb{R}^2$ is compact and $\mathrm{Fav}(E \cap B(x,r)) \geq \kappa r$ for all $x \in E$, then

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**Quantitative Vitushkin's conjecture**

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# Thank you!