Favard length and quantitative rectifiability

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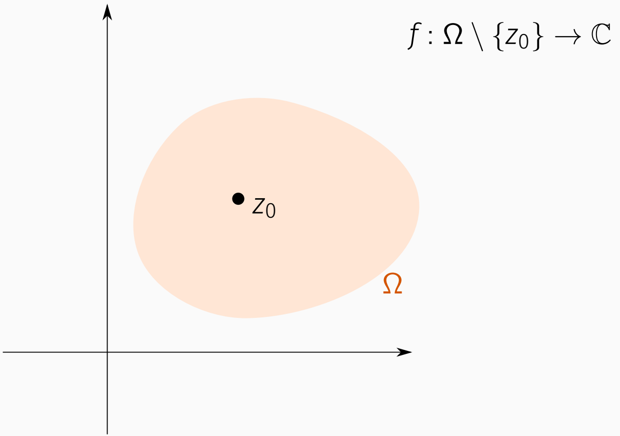
Vitushkin’s conjecture

Riemann's theorem on removable singularities

Theorem (Riemann)

If $z_0 \in \Omega \subset \mathbb{C}$ and $f : \Omega \setminus \{z_0\} \to \mathbb{C}$ is analytic and bounded, then f can by extended analytically to all of Ω.



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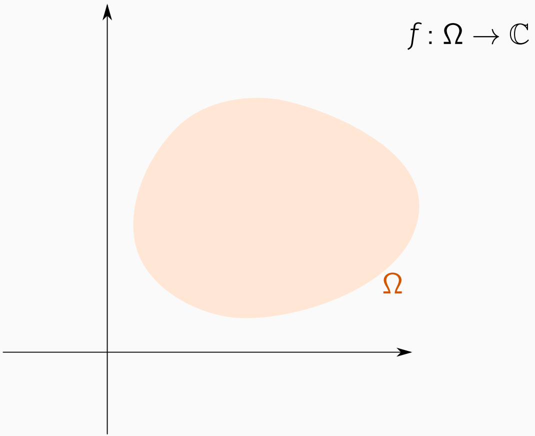
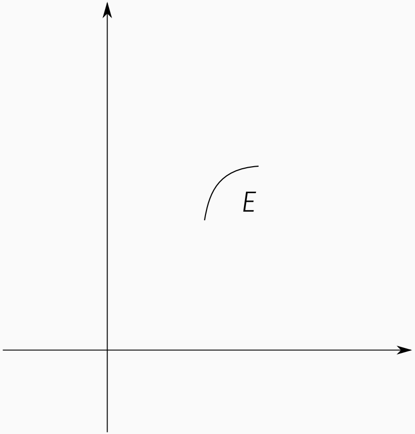


Removable sets

A compact set $E \subset \mathbb{C}$ is **removable for bounded analytic functions** if for any open $\Omega \subset \mathbb{C}$ containing E, each bounded analytic function $f : \Omega \setminus E \to \mathbb{C}$ has an analytic extension to Ω.



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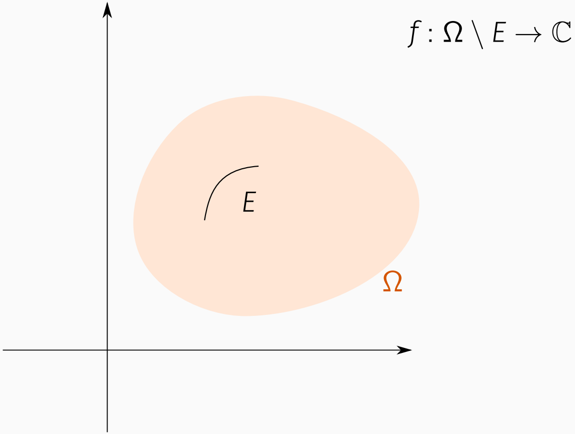

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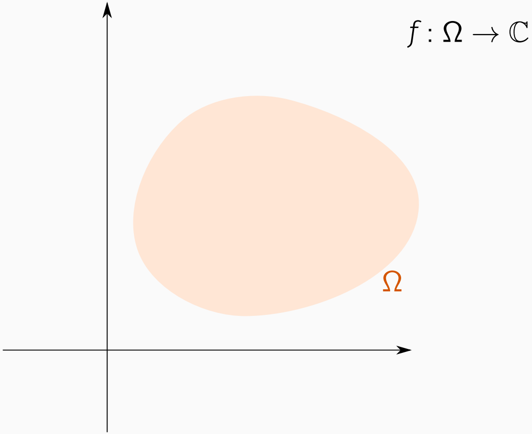

Analytic capacity

In 1947 Ahlfors characterized removability in terms of **analytic capacity**:

$$E \text{ is removable} \quad \Leftrightarrow \quad \gamma(E) = 0,$$

where

$$\gamma(E) = \sup\{|f'(\infty)| \;:\; f : \mathbb{C} \setminus E \to \mathbb{C} \text{ analytic}, \; \|f\|_\infty \le 1\},$$
$$f'(\infty) = \lim_{z \to \infty} z(f(z) - f(\infty)).$$

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Find a geometric characterization of removable compact sets, i.e. compact sets with $\gamma(E) = 0$.

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- If $\dim_H(E) > 1$, then $\gamma(E) > 0$.

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Question

$\gamma(E) = 0 \quad \Leftrightarrow \quad \mathcal{H}^1(E) = 0$? No!

There are sets $E \subset \mathbb{C}$ with $\gamma(E) = 0$ and $0 < \mathcal{H}^1(E) < \infty$.
(Vitushkin 1959, Garnett, Ivanov 1970s)

Vitushkin's conjecture

The sets constructed by Vitushkin, Garnett and Ivanov had very small projections. More precisely, they satisfied

$$\mathcal{H}^1(\pi_\theta(E)) = 0$$

for a.e. direction $\theta \in [0, \pi]$.

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Define Favard length of E as

$$\mathrm{Fav}(E) = \int_0^\pi \mathcal{H}^1(\pi_\theta(E))\, d\theta.$$

Vitushkin's conjecture

$$\gamma(E) = 0 \quad \Leftrightarrow \quad \mathrm{Fav}(E) = 0$$

Solution to Vitushkin's conjecture

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$$\gamma(E) = 0 \quad \Leftrightarrow \quad \mathrm{Fav}(E) = 0$$

- In the case $\mathcal{H}^1(E) < \infty$ Vitushkin's conjecture is **true**! (Calderón '77, David '98)

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$$\mathrm{Fav}(E) = 0 \quad \not\Rightarrow \quad \gamma(E) = 0.$$

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- What about

$$\mathrm{Fav}(E) = 0 \quad \Leftarrow \quad \gamma(E) = 0?$$

Open problems

Problem 1 (qualitative)

$$\mathrm{Fav}(E) > 0 \quad \Rightarrow \quad \gamma(E) > 0?$$

Open for sets $E \subset \mathbb{C}$ with $\dim_H(E) = 1$ and non-σ-finite $\mathcal{H}^1$-measure.

Open problems

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Problem 2 (quantitative)

$$\gamma(E) \gtrsim \mathrm{Fav}(E)?$$

$$\gamma(E) \gtrsim_{\mathrm{Fav}(E)} 1?$$

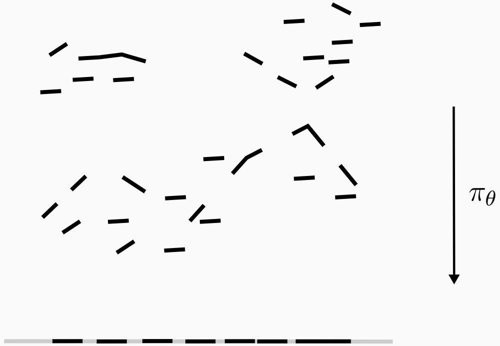
Open even for sets with finite length.

What happens for sets with finite length?

Two ingredients

Theorem (Besicovitch 1939)

Let $E \subset \mathbb{R}^2$ with $0 < \mathcal{H}^1(E) < \infty$. If $\mathrm{Fav}(E) > 0$,



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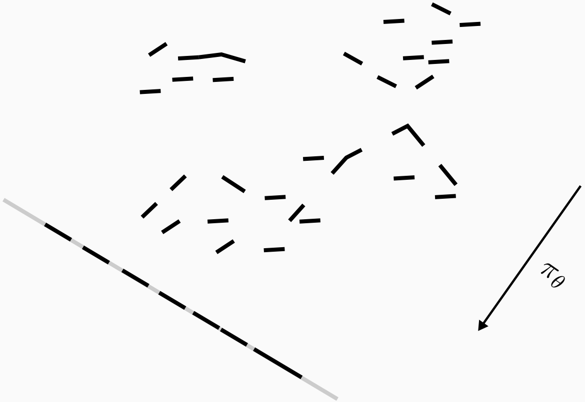

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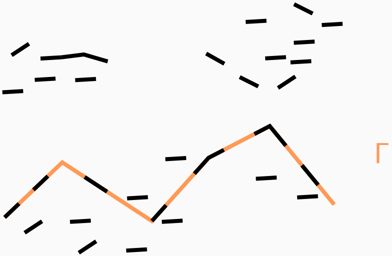
Theorem (Besicovitch 1939)

Let $E \subset \mathbb{R}^2$ with $0 < \mathcal{H}^1(E) < \infty$. If $\mathrm{Fav}(E) > 0$, then there exists a Lipschitz graph $\Gamma \subset \mathbb{R}^2$ with $\mathcal{H}^1(E \cap \Gamma) > 0$.



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Theorem (Calderón 1977)

If Γ is a rectifiable curve and $F \subset \Gamma$ satisfies $\mathcal{H}^1(F) > 0$, then

$$\gamma(F) > 0.$$

This is a corollary of Calderón's proof of the L^2-boundedness of Cauchy transform on Lipschitz graphs with small constant.

Vitushkin's conjecture when $\mathcal{H}^1(E) < \infty$

Goal

$$\mathrm{Fav}(E) > 0 \quad \Rightarrow \quad \gamma(E) > 0$$

If $0 < \mathcal{H}^1(E) < \infty$ and $\mathrm{Fav}(E) > 0$, then by the Besicovitch projection theorem $\exists\, \Gamma$ with $\mathcal{H}^1(E \cap \Gamma) > 0$

$$\gamma(E) \geq \gamma(E \cap \Gamma)$$

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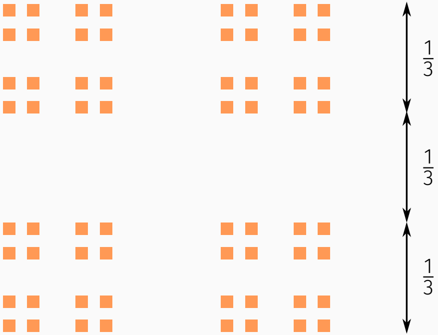
∎

- Why does it only work for sets with finite length?
- Why does it give no quantitative estimates?

First problem

The Besicovitch projection theorem **fails** for sets with infinite length!



Set $K = C_{1/3} \times C_{1/3}$ satisfies $\mathrm{Fav}(K) \gtrsim 1$ and $\mathcal{H}^1(K \cap \Gamma) = 0$ for every rectifiable curve.

Second problem

Recall: if $0 < \mathcal{H}^1(E) < \infty$ and $\mathrm{Fav}(E) > 0$, then $\exists\, \Gamma$ with $\mathcal{H}^1(E \cap \Gamma) > 0$ and

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There are estimates on $\gamma(E \cap \Gamma)$ depending on $\mathcal{H}^1(E \cap \Gamma)$, e.g. if Γ is an L-Lipschitz graph, then

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Favard length problem

Can we quantify the dependence of $\mathrm{Lip}(\Gamma)$ and $\mathcal{H}^1(E \cap \Gamma)$ on $\mathrm{Fav}(E)$?

Favard length problem

Naive conjecture...

Theorem (Besicovitch 1939)

Let $E \subset \mathbb{R}^2$ with $0 < \mathcal{H}^1(E) < \infty$. If $\mathrm{Fav}(E) > 0$, then there exists a Lipschitz graph $\Gamma \subset \mathbb{R}^2$ with

$$\mathcal{H}^1(E \cap \Gamma) > 0.$$

Naive conjecture

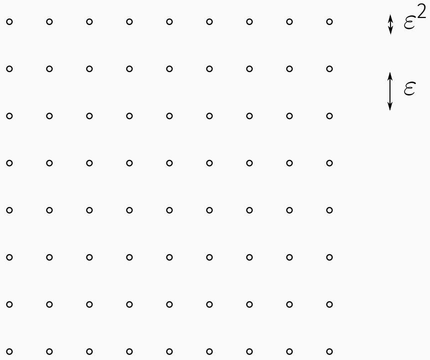
Let $E \subset [0,1]^2$ with $\mathcal{H}^1(E) \sim 1$ and $\mathrm{Fav}(E) \gtrsim 1$. Then, there exists a Lipschitz graph $\Gamma \subset \mathbb{R}^2$ with $\mathrm{Lip}(\Gamma) \lesssim 1$ and

$$\mathcal{H}^1(E \cap \Gamma) \gtrsim 1.$$

… is false

For any $\varepsilon > 0$ there exists a set $E = E_\varepsilon \subset [0,1]^2$ with $\mathcal{H}^1(E) \sim 1$ and $\mathrm{Fav}(E) \gtrsim 1$ such that for all L-Lipschitz graphs Γ

$$\mathcal{H}^1(E \cap \Gamma) \lesssim L\varepsilon.$$

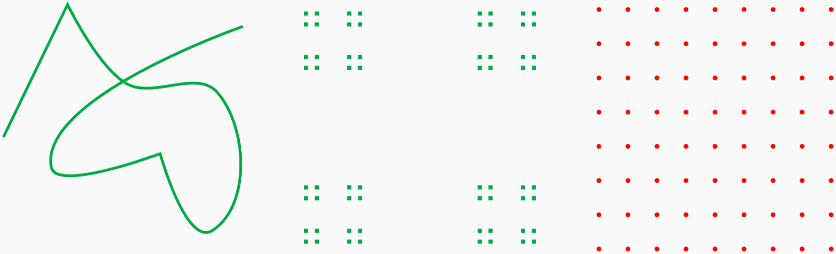


E consists of ε^{-2} uniformly distributed circles of radius ε^2.

Reasonable conjecture

We say that $E \subset \mathbb{R}^2$ is **Ahlfors regular** if for every $x \in E$ and $0 < r < \text{diam}(E)$

$$C^{-1} r \leq \mathcal{H}^1(E \cap B(x,r)) \leq Cr.$$



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Reasonable conjecture

Let $E \subset \mathbb{R}^2$ be an Ahlfors regular set with $\mathrm{Fav}(E) \gtrsim \mathcal{H}^1(E)$.

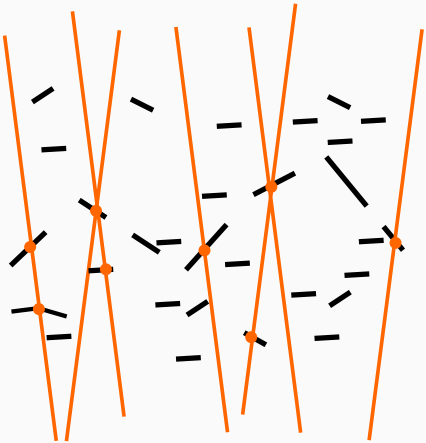
Then, there exists a Lipschitz graph $\Gamma \subset \mathbb{R}^2$ with $\mathrm{Lip}(\Gamma) \lesssim 1$ and

$$\mathcal{H}^1(E \cap \Gamma) \gtrsim \mathcal{H}^1(E).$$

Variations on this conjecture appearing since the 90s in the works of David and Semmes, Mattila, Peres and Solomyak.

What is this really about?

many lines with few intersections



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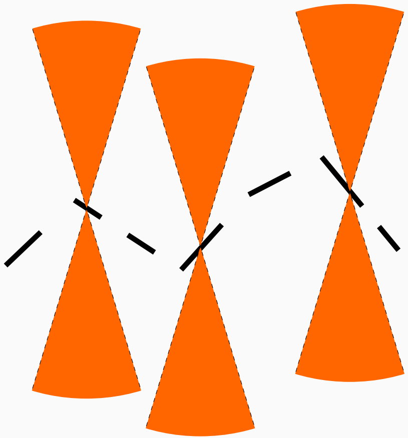

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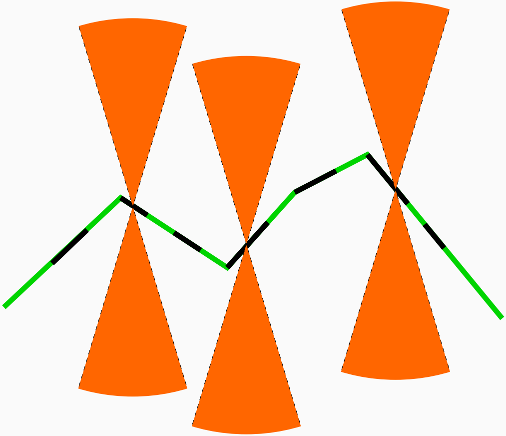
$\Rightarrow$ cones with no intersections

$\Rightarrow$ subset of a Lipschitz graph



Previous work

Reasonable conjecture

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Then, there exists a Lipschitz graph $\Gamma \subset \mathbb{R}^2$ with $\mathrm{Lip}(\Gamma) \lesssim 1$ and

$$\mathcal{H}^1(E \cap \Gamma) \gtrsim \mathcal{H}^1(E).$$

Progress on the conjecture consisted of replacing "$\mathrm{Fav}(E) \gtrsim \mathcal{H}^1(E)$" by:

- **David-Semmes '93**: big projection + WGL
- **Martikainen-Orponen '18**: projections in L^2
- **Orponen '21**: plenty of big projections
- **D. '22**: projections in L^∞

New result: the conjecture is true!

Theorem (D. '24)

Let $E \subset \mathbb{R}^2$ be an Ahlfors regular set with $\mathsf{Fav}(E) \geq \kappa \mathcal{H}^1(E)$.

Then, there exists a Lipschitz graph $\Gamma \subset \mathbb{R}^2$ with $\mathsf{Lip}(\Gamma) \lesssim_\kappa 1$ and

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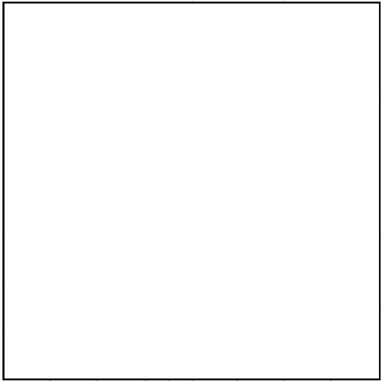
Corollaries:

- a positive answer to a 1993 question of David and Semmes,
- a positive answer to a 2002 question of Peres and Solomyak,
- progress on Vitushkin's conjecture.

About the proof

- main tool: conical energies of [Chang-Tolsa ’20]
- uses [Martikainen-Orponen ’18] as a black-box
- key novelty: multiscale decomposition involving scales, locations, and directions:



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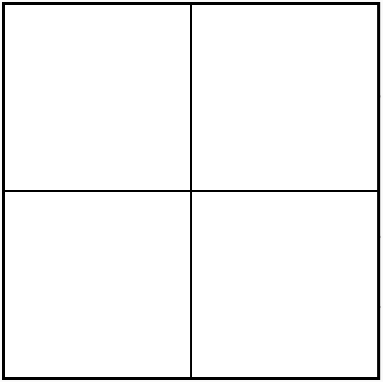

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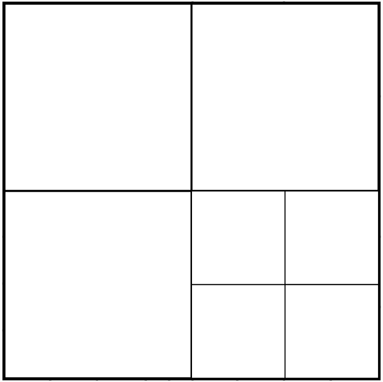

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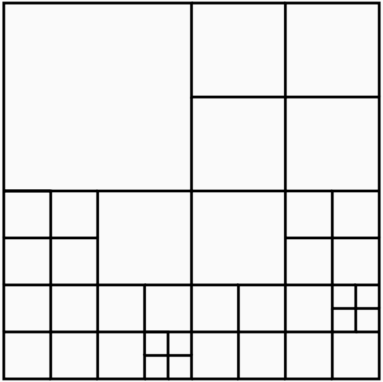

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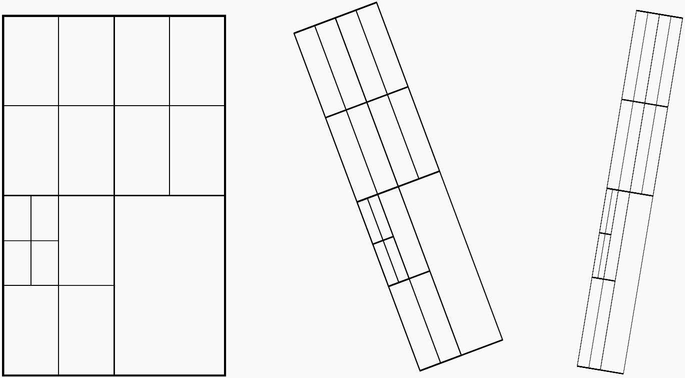

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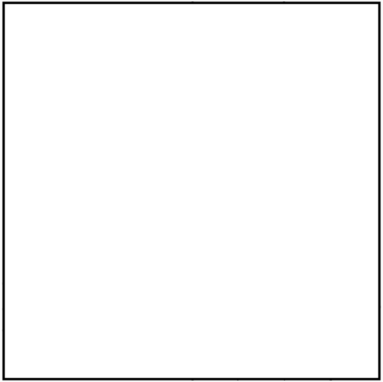

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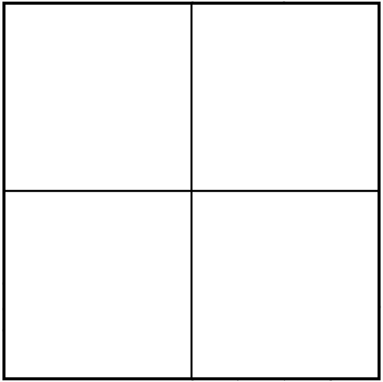

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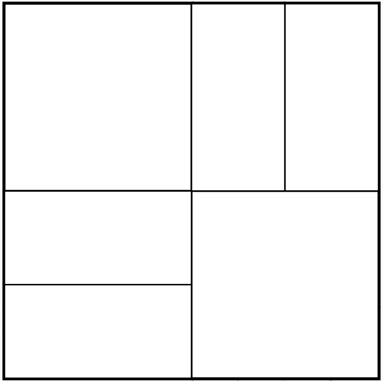

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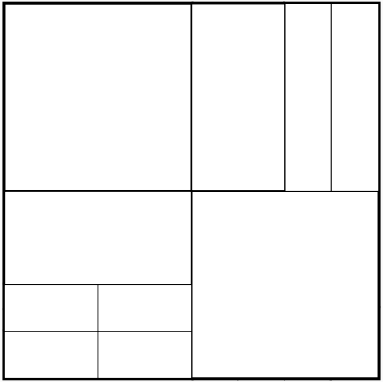

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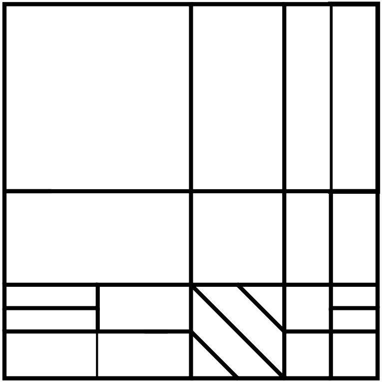

Back to Vitushkin

Estimates for Ahlfors regular sets

Quantitative Vitushkin's conjecture

If $E \subset \mathbb{R}^2$ is compact and $\mathsf{Fav}(E) \geq \kappa\, \mathsf{diam}(E)$, do we have

$$\gamma(E) \gtrsim_\kappa \mathsf{diam}(E)?$$

Partial results in Chang-Tolsa '20 and D.-Villa '22.

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Corollary (D. '24)

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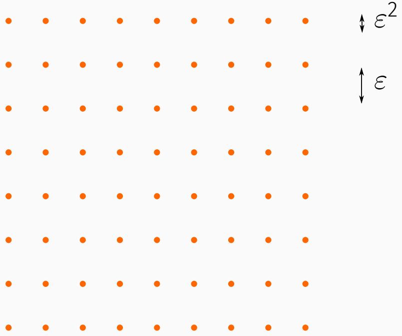

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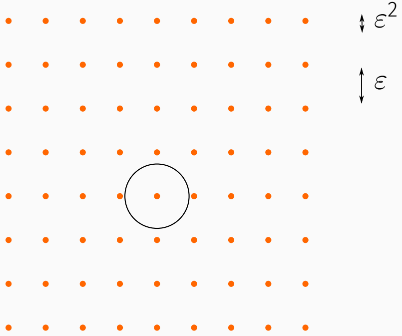

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If $E \subset \mathbb{R}^2$ has ULFL, then

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Thank you!